On the rate boundary value problem for damage modelization by thick level-set

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ON THE RATE BOUNDARY VALUE PROBLEM FOR DAMAGE MODELIZATION BY THICK LEVEL-SET

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ABSTRACT

In total damage the rupture occurs on a moving surface along which strong discontinuties of displacement gradient are localized. A damage modelisation is proposed based on a continuous transition from undamaged to damaged material. In this new framework, the evolution of damage is associated with a moving layer of finite length l_c. With this description, initiation and propagation of damage can be unified in the same constitutive law. Using a normality law based on the driving force associated with the motion of the layer, the solution of the rate boundary value problem of propagation and displacement satisfies a variational inequality. Characterization of uniqueness is then given.

Key words: *Damage, Level-set, Moving surface, Moving layer, XFEM*

1. INTRODUCTION

Generally, fracture mechanics is not adapted to model the degradation of solids under mechanical loading. The initiation of defect requires damage modelization for describing the gradual loss of local stiffness.

Severall approaches are usefull to describe such situation. For instance, constitutive laws based on second gradient description of damage [1,2] or phase field [3,4] have been recently proposed for this purpose. In this paper we propose to study the model of damage proposed recently using of level-set approach [5].

For elastic quasi-brittle material, the evolution of the interface separating the undamaged material $d = 0$ to the damaged one $d = 1$ have been studied in a framework based on an energetical description [6,7]. In total brittle damage, the damaged zone can not support any further tension after some critical prescribed value in stress, strain, or free energy. With this property, the dissipation is obtained by an integral along a moving surface [6] where the driving force $G(s)$ has the form of an energy release rate. When the velocity of propagation ($a(s)$) of the damaged front is governed by a normality rule based on this driving force, many variational formulations of the rate boundary value problems have been established [6,7,8]. The rate boundary value problem for brittle material can have multiplicity of solutions when the propagation law is governed by a generalized Griffith's law:

$$a(s) \geq 0, \quad G(s) \leq G_c, \quad (G(s) - G_c)a(s) = 0. \tag{1}$$

This model has no characteristic length. Moreover it has been shown that taking into account of surface energy along the damaged front plays a role on the uniqueness of the velocity $a(s)$. In this case the front is more stable [9] because the new driving force depends now on the mean curvature κ_m of the moving front:

$$G_\beta(s) = G(s) - \kappa_m(s)\beta \leq G_c, \tag{2}$$

Using this framework, the propagation of an existing interface between an undamaged and a damaged zone is studied. The presence of surface energy density β stabilizes the propagation of the interface. This interface has no thickness and mechanical quantities present discontinuities.

To avoid these discontinuities, a new approach is proposed based on the propagation of a moving layer inside which the damage is a continuous function of the position. The evolution of damage is then associated to the motion of a layer of finite length [5].

The initial material and the damaged material are separated by a surface Γ. This boundary is a moving interface. A surface is an isopotential or a level-set. Through the interface the material changes its mechanical properties. In the proposed description, this transition is continuous.

2. THE MODEL OF DAMAGE

We consider a body Ω under tension $\underline{T}^d$ over $\partial\Omega_T$ and prescribed displacement $\underline{u}^d$ on the complementary part of the boundary $\partial\Omega_u$. Under this loading, the body is deformed and a displacement field $\underline{u}$ described the motion of all material points of the body.

The material of the body has an elastic behaviour with moduli evolving with damage. The free energy of the body $w(\boldsymbol{\varepsilon}, d)$ is a function of the strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\underline{u}^T + \nabla\underline{u})$ and of a scalar damage variable $d, 0 \leq d \leq 1$.

The state equations are defined classically as:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, Y = -\frac{\partial w}{\partial d}, \tag{3}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. The mechanism of dissipation is only due to damage and the dissipation of the whole body is reduced to

$$D_m = \int_\Omega Y\dot{d}\;\mathrm{d}\Omega \geq 0. \tag{4}$$

When damage is established the whole body is decomposed in three parts, the undamaged body Ω_1, the transition zone Ω_c (where $0 < d < 1$) and the damaged material Ω_2 where

($d = 1$). On the boundary $\partial\Omega_c$ the free energy is continuous, there is no discontinuties of the stress vector and the moduli of elasticity are continuous. Then when the layer Ω_c is moving, there exists no dissipation along the boundary of the layer. A more detail discussion is given in [5].

The level-set $\phi = 0$ gives the position of Γ the part of boundary Ω_c where $d = 0$. We assume that the damage d is a continuous explicit function $d(\phi)$ of the distance to the surface Γ.

In the domain where the gradient of the level-set is continuous, the damage is defined by

$$d = 0,\ \phi \leq 0 \quad ; \quad d'(\phi) \geq 0,\ 0 \leq \phi < l_c \quad ; \quad d(\phi) = 1,\ \phi \geq l_c. \tag{5}$$

Then the surface $d(X, t) = d_o$ is also a level-set. This representation of damage is illustrated Figure 1. The minimum length separating the level-set $d = 0$ to the level-set $d = 1$ is l_c.

The description of the behaviour of the system is related to the motion of a layer with thickness $l \leq l_c$.

3. ON THE MOTION OF A LAYER

We study the motion of a thick layer. The study is made for plane motion to simplify the expression. The curve Γ is the interface separating the undamaged material to the damaged one. A point M_{t} of Γ is referred by its curvilinear coordinates s, its position is $\underline{X}_o(s)$. The local frame is then defined by the tangential vector $\underline{T} = \dfrac{\mathrm{d}\underline{X}_o}{\mathrm{d}s}$. The normal vector $\underline{N}$ satisfies the Fresnet relation

$$\frac{\mathrm{d}\underline{T}}{\mathrm{d}s} = \kappa \underline{N}, \quad \frac{\mathrm{d}\underline{N}}{\mathrm{d}s} = -\kappa \underline{T} \tag{6}$$

where κ is the curvature of the curve Γ_o at point $\underline{X}_o$.

A point M of the layer has coordinates (s, z) in the frame $(\underline{T}, \underline{N})$,

$$\underline{X} = \underline{X}_o + z\underline{N} \tag{7}$$

then the local frame at $\underline{X}$ is defined by

$$\mathrm{d}\underline{X} = \ \mathrm{d}s\underline{\tau} + \ \mathrm{d}z\underline{\nu} \tag{8}$$

and depends on the position inside the layer,

$$\underline{\tau} = (1 - \kappa z)\, \underline{T}, \quad \underline{\nu} = \underline{N} \tag{9}$$

The level-set $\phi(\underline{X}_o, t) = 0$ is the curve Γ, during the motion the local frame $(\underline{T}, \underline{N})$ is changing. We know that $\underline{N} = \dfrac{\partial \phi}{\partial \underline{X}} / || \dfrac{\partial \phi}{\partial \underline{X}} ||$, and

$$\frac{\partial \phi}{\partial \underline{X}} . \underline{\phi} + \frac{\partial \phi}{\partial \mathrm{t}} = 0, \tag{10}$$

this defines the normal velocity $a(s)$: $\underline{\phi} = a(s)\underline{N}$, The same is true for all level set $\phi(\underline{X}, t) = z$.

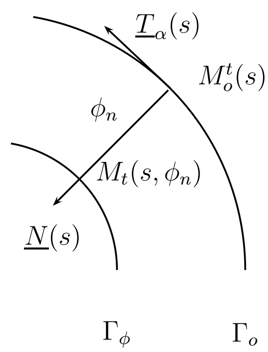

Figure 1: The local frame

Actual geometry and convected geometry. The actual position $\underline{X}_o^{\mathrm{t}}$ of a point of Γ satisfies the equation of motion $\underline{X}_o^{\mathrm{t+dt}} = \underline{X}_o^{\mathrm{t}} + \underline{\phi}(s)\mathrm{dt}$, the evolution of the local frame is then deduced. For any geometrical quantity G as $\underline{T},\ \underline{N},\ \kappa$, we can define the derivative following the motion of the surface Γ by

$$D_a G = \lim_{\mathrm{dt}\to 0} \frac{G_{\mathrm{t+dt}} - G_{\mathrm{t}}}{\mathrm{dt}}.$$

and we obtain

$$D_\phi \underline{T} = \frac{\mathrm{d}\underline{\phi}}{\mathrm{d}S}.\underline{N}\ \ \underline{N}, \quad D_\phi \underline{N} = -\frac{\mathrm{d}\underline{\phi}}{\mathrm{d}S}.\underline{N}\ \ \underline{T}, \quad D_\phi \kappa = \underline{N}.\frac{\mathrm{d}^2\underline{\phi}}{\mathrm{d}S^2} - 2\frac{\mathrm{d}\underline{\phi}}{\mathrm{d}S}.\underline{T}\ .$$

A point $\underline{X} = \underline{X}_o + z\underline{N}$ of the layer is on the level-set $\phi(\underline{X}, t) = z$. At time $\mathrm{t} + \mathrm{dt}$, the actual position is $\underline{X}^{\mathrm{t+dt}}$ such that

$$\underline{X}^{\mathrm{t+dt}} = \underline{X}_o^{\mathrm{t+dt}} + z\ \underline{N}^{\mathrm{t+dt}} \tag{11}$$

then the evolution of $\underline{X}$ is given by,

$$D_a \underline{X} = \lim_{\mathrm{dt}\to 0} \frac{(\underline{x} - \underline{X})}{\mathrm{dt}} = a(s)\underline{N} - z\ \frac{\mathrm{d}a}{\mathrm{d}S}\underline{T} \tag{12}$$

At point $\underline{X}$ the variation of any mechanical quantities $f(\underline{X}, t)$ following the motion of the layer is then

$$D_a f = lim_{\eta\to 0} \frac{f(\underline{X} + \eta D_a \underline{X}, \mathrm{t} + \eta) - f(\underline{X}, \mathrm{t})}{\eta} = \frac{\partial f}{\partial \mathrm{t}} + \frac{\partial f}{\partial \underline{X}}.D_a\underline{X} \tag{13}$$

For the function $\phi(\underline{X}, \mathrm{t}) = z$, we have $\nabla\phi = \underline{N}$, $\Delta\phi = -\dfrac{\kappa}{1 - z\kappa}$ and $D_a\ \mathrm{d}S = -\kappa a\ \mathrm{d}S$. These definitions are usefull to characterize the fact that the damage variable d is a continuous function of $\phi(\underline{X}, t) = z$.

Variations of averaged quantity on the layer. To study the evolution of the driving force associated to the motion of the layer, we must study the evolution of quantities such as

$$F = \int_o^l f(1 - z\kappa)\mathrm{d}z \ \bar{F} = \int_\Gamma F \ \mathrm{d}S, \tag{14}$$

we obtain

$$\begin{aligned} D_a\bar{F} &= D_a \int_{\Omega_c} f \ \mathrm{d}\Omega = \int_\Gamma D_a F - a(s)\kappa F \ \mathrm{d}S & (15) \\ D_a F &= \int_o^l (1 - z\kappa) D_a f \ \mathrm{d}z - \int_o^l z f D_a \kappa \ dz & (16) \end{aligned}$$

The dissipation of the system. With this definition, the dissipation is obtained as

$$D_m = \int_\Gamma \int_o^l Y d'(\phi) \ \ (1 - \kappa \ \phi) \ \dot{\phi} \ d\phi \ \mathrm{d}S. \tag{17}$$

The evolution of the level-set is given by the evolution of the moving surface $\phi(X, t) = z$ then

$$\dot{\phi} - a(s)\nabla\phi.\underline{N} = 0, \tag{18}$$

where the velocity a is the normal speed of the iso-ϕ and $\underline{N} = \nabla\phi/||\nabla\phi||$ is the normal vector to the surface $\phi = z$.

The driving force associated to the velocity a is given by the motion of the layer according to the dissipation

$$D_m = \int_\Gamma G(s)a(s) \ \mathrm{d}S, \text{where } G(s) = \int_o^l Y\nabla d.\underline{N} \ \ det(1 - \kappa\phi)\mathrm{d}\phi. \tag{19}$$

The curvature plays now a role in the expression of the dissipation.

The velocity a is determined with respect to a constitutive law based on the driving force $G(s)$. As in previous paper [10], we propose to consider a generalized Griffith's law for $(l(s, t) \leq l_c)$

$$a(s) \geq 0, \quad G(s) \leq \bar{G}(s) = \int_o^l Y_c\nabla d.\underline{N} \ \ (1 - \kappa\phi)d\phi, \quad (G(s) - \bar{G}(s))a(s) = 0, \tag{20}$$

which is an averaged yielding function on the layer. This generalizes the normality rule proposed for a sharp interface. Now, the damage in the layer is continuous with a given gradient, this is a model of continuum with graded damage.

The definition of the driving force (eq.19) and the normality (eq.20) ensures the positivity of entropy production.

4. A MODEL OF BAR WITH A MOVING LAYER

Analysis of the system. The free energy w for unixial response is

$$w(\boldsymbol{\varepsilon}, d) = \frac{1}{2}E(d)\boldsymbol{\varepsilon}^2, \quad Y = -\frac{\partial w}{\partial d}, \tag{21}$$

where d varies from 0 to 1, the Young modulus $E(d)$ is a continuous fonction of d, then there is no discontinuity at $d = 0$. For comparison with a sharp interface we consider the matching conditions $E(0) = E_1$ and $E(1) = E_2$.

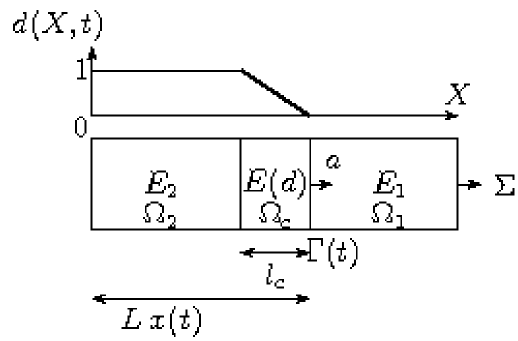

Figure 2: The propagation of a layer

On Figure 2 the value of the damage parameter is given by ($\phi(X, t) = \Gamma(t) - X$)

$$\begin{cases} \phi(X,t) \leq 0 & d(X,t) = 0, \\ 0 \leq \phi(X,t) \leq l_c & d(X,t) = \phi/l_c, \\ \phi(X,t) \geq l_c & d(X,t) = 1. \end{cases} \tag{22}$$

The damage parameter d is an increasing function of the distance ϕ to the boundary Γ separating the sound material to the damaged one. The function $d(\phi)$ is a given continuous function of ϕ

Initially $\Gamma(t) = 0$ and the propagation of the layer begins at the origin of the bar, so the thickness $\Gamma(t) = l(t)$ is smaller than l_c. The thickness increases to l_c and after this step of initialization, the thickness is kept constant.

For the given constitutive laws, the dissipation is local and only due to damage $d_m = Y\dot{d}$. From the integration over the layer we get the total dissipation due to damage inside the bar:

$$D_m = \int_o^l Y\dot{d}\, \mathrm{d}\phi. \tag{23}$$

Assuming that d is a continuous function of ϕ, then E becomes a continuous function of ϕ. The prime denotes the derivative with respect to ϕ, $d'(\phi) = \dfrac{\mathrm{d}d}{\mathrm{d}\phi}$ and $E'(\phi) = \dfrac{\mathrm{d}E}{\mathrm{d}\phi}$. To define the local force Y (eq.3) we need the derivative $\dfrac{\mathrm{d}E}{\mathrm{d}d}$

$$\frac{\mathrm{d}E}{\mathrm{d}d} = \frac{\mathrm{d}E}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}d}. \tag{24}$$

The fact that this derivative must be finite implies properties on $E'(\phi), d'(\phi)$. Moreover, the local force Y is

$$Y = -\frac{E'(\phi)}{2d'(\phi)}\boldsymbol{\varepsilon}^2 = -\frac{E'(\phi)}{2E^2 d'(\phi)}\Sigma^2. \tag{25}$$

As the velocity ϕ satisfies $\dot{\phi} = a(s)$, the total dissipation is finally expressed as

$$D_m = \frac{a}{2}\int_o^l \Sigma^2(-\frac{E'(\phi)}{E^2(\phi)})\mathrm{d}\phi = G(l,\Sigma)a, \tag{26}$$

where $G(l,\Sigma)$ is

$$G(l,\Sigma) = \frac{1}{2}\Sigma^2(\frac{1}{E(l/l_c)} - \frac{1}{E(0)}). \tag{27}$$

When $l = l_c$, we recover the expression obtained for a sharp interface, for which the dissipation is $D_m = G_c\ a$. In this case, the strain $\boldsymbol{\varepsilon}$ and the moduli of elasticity are discontinuous. The total energy is given by ($x = \Gamma/L, \dot{x} = a\ l$):

$$W = \frac{1}{2}(\frac{x}{E_1} + \frac{1-x}{E_2})L\Sigma^2 \tag{28}$$

and the dissipation is given by

$$D_m = -\frac{\partial W}{\partial x}\dot{x} = \frac{1}{2}(\frac{1}{E_2} - \frac{1}{E_1})\Sigma^2 a \tag{29}$$

So when the layer is established, the dissipations described by a sharp interface or by a moving layer are identical.

If ϕ vanishes the limit value $Y(0^+,\Sigma)$ is

$$Y(0^+,\Sigma) = -\frac{1}{2}\frac{\Sigma^2}{E_1^2}\lim_{\phi\to 0}\frac{E'(\phi)}{d'(\phi)}. \tag{30}$$

When we adopt the normality rule (eq.20), the value of Y must be smaller than Y_c. This defines the critical value for initiation of damage in a point of the bar. From (eq.30), the corresponding critical value of Σ is Σ_o such that

$$-\frac{1}{2}\frac{\Sigma_o^2}{E_1^2}\lim_{\phi\to 0}\frac{E'(\phi)}{d'(\phi)} = Y_c \tag{31}$$

It can be noticed that the critical value Σ_o depends of the damage law and is generally greater from Σ_c. We assume that the dissipation of the system is the same when the layer moves with the limit thickness l_c this gives a relation between the value Y_c and G_c.

$$D_m = G_c a = Y_c d(l_c) a \tag{32}$$

5. ON THE RATE BOUNDARY VALUE PROBLEM

At time t the actual position of a material point is defined by the displacement $\underline{u}$, the position of the layer Γ, $l(s)$ is known and the solution is inside the domain of reversibility.

$$F = \int_o^l Y d'(\phi)(1-\kappa z)\mathrm{d}z - Y_c \int_o^l d'(\phi)(1-\kappa\ z)\mathrm{d}z \leq 0. \tag{33}$$

The evolution is governed by

$$a(s) \geq 0, \quad F \leq 0, \quad a(s)F = 0 \tag{34}$$

It is obvious that $a(s)$ is positive if $F = 0$. At this state F satisfies $\dot{F} \leq 0$. The derivation of the consistency condition $aF = 0$ implies that $a(s) > 0$ if $\dot{F} = 0$. Then the set of admissible field $a(s)$ satisfies

$$\int_\Gamma (a(s) - a^*(s)) D_a F \ \mathrm{d}s \geq 0 \tag{35}$$

this is a variational inequality to solve on the set of admissible fields :

$$a^*(s) \geq 0, \text{ along } \Gamma^+ = \{s \in \Gamma / F(s) = 0\}. \tag{36}$$

We must explain the variations of F following the motion of the layer $D_a F$ where F is defined by (eq.33):

$$D_a \int_o^l f(1-\kappa z)dz = \int_o^l D_a f(1-\kappa z)dz - \int_o^l z f D_a \kappa \mathrm{d}z \tag{37}$$

For $f(\boldsymbol{\varepsilon}, \phi) = Y d'(\phi)$ we have the property

$$D_a \phi = 0, \quad D_a f(\boldsymbol{\varepsilon}, \phi) = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} D_a \boldsymbol{\varepsilon} = \dot{f} + a \frac{\partial f}{\partial z} - \frac{z}{1 - z\kappa} \frac{\mathrm{d}a}{\mathrm{d}s} \frac{\partial f}{\partial s} \tag{38}$$

After simplification and integration by parts along Γ we obtain

$$\begin{aligned} 0 \ \leq \ & \int_\Gamma (a - a^*) \Big[\int_o^l \dot{f}(1-\kappa z)\mathrm{d}z + \int_o^l a \Big(\frac{\partial f}{\partial z}(1-\kappa z) - z f \kappa^2 \Big) \mathrm{d}z \Big] \ \mathrm{d}s \\ - \ & \int_\Gamma (\int_o^l f \mathrm{d}z) \frac{\mathrm{d}a}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}s}(a - a^*) \ \mathrm{d}s \end{aligned} \tag{39}$$

As $\dot{f} = \dfrac{\partial f}{\partial \boldsymbol{\varepsilon}} \dot{\boldsymbol{\varepsilon}} - a \dfrac{\partial f}{\partial \phi}$ the inequality becomes

$$0 \leq \int_\Gamma (a - a^*) \Big(\int_o^l \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}(1-\kappa z) \ \mathrm{d}z \Big) \ \mathrm{d}s - \int_\Gamma (a - a^*) L a \ \mathrm{d}s - \int_\Gamma \frac{\mathrm{d}a}{\mathrm{d}s} M \frac{\mathrm{d}}{\mathrm{d}s}(a - a^*) \ \mathrm{d}s$$

where L, M are functions of the actual state

$$L = -\int_o^l \Big(\frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \frac{\partial \boldsymbol{\varepsilon}}{\partial z}(1-\kappa z) - z f \kappa^2 \Big) \ \mathrm{d}z \quad M = -\int_o^l f z \ \mathrm{d}z \tag{40}$$

Property of the solution. The solution of the rate boundary value problem satisfies the variational inequality

$$\frac{\partial \mathcal{F}}{\partial \underline{v}}.(\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial a}(a - a^*) \leq 0 \tag{41}$$

where

$$\begin{aligned} \mathcal{F} &= \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}(\underline{v}) : C(d) : \boldsymbol{\varepsilon}(\underline{v}) \,\mathrm{d}\Omega - \int_\Gamma a \int_o^l \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v})(1 - \kappa z) \,\mathrm{d}z \,\mathrm{d}s \\ &+ \int_\Gamma \frac{1}{2}(a^2 M + L(\frac{\mathrm{d}a}{\mathrm{d}s})^2) \,\mathrm{d}s \end{aligned}$$

The proof is easy to obtain. Studying the properties of the operator $\mathcal{F}$ give conditions on stability and uniqueness of velocity $a(s)$ as proposed in [6,9]. It can be observed that the presence of a and da/ds in the functional give a non local contribution along the curve Γ, this have a strong influence on the regularity of a. This variational inequality is extended without strong difficulties to 3D, the main point is to introduce a curvature tensor for the surface Γ and to generalize the derivation of any quantity following the motion of Γ.

We study now the evolution of a cylinder under radial extension.

The response of a cylinder. We consider a cylinder with external radius R_e in plane strain. For analytical treatment we consider that the shear modulus is constant and the damage parameter governs the evolution of the bulk modulus. In this case, the Lamé coefficient λ is a function of d. The solution for isotropic elasticity is given by the radial displacement $\underline{u} = u(R)\underline{e}_r$. The solution of the problem of linear elasticity is

$$R^2 u(R) = A \int_o^R \frac{rdr}{L(r)}, \quad L = \lambda + 2\mu \tag{42}$$

The constant A is determined by the prescribed displacement at point $R = R_e : u(R_e) = ER_e$, where E is a increasing function, then

$$R_e E = AK = A \int_o^{R_e} \frac{rdr}{L(r)} \tag{43}$$

Initially, the body is homogeneous with characteristic $\lambda_1 = \lambda_o, L_1 = \lambda_1 + 2\mu$, and when $d = 0$ the value os the constant is $A = (\lambda_o + 2\mu)2E$, the total energy is $W = 2\pi R_e^2 2(\lambda_o + \mu)E^2$. For some critical value of E the damage initiates, and λ is no more uniform. We assume for example that

$$\frac{1}{L(\phi)} = \frac{1 - \phi/l_c}{\lambda_1 + 2\mu} + \frac{\phi/l_c}{\lambda_2 + 2\mu} \tag{44}$$

$$d(r) = \phi/l_c \tag{45}$$

During the initiation of the layer, the position γ of the interface Γ determines the value of the level set $\phi = \gamma - R$, then

$$K(\gamma) = \int_o^{R_e} \frac{rdr}{L} = \frac{R_e^2}{2L_1} + \frac{\gamma^3}{6l_c}(\frac{1}{L_2 - L_1}), \quad L_2 = \lambda_1 + 2\mu. \tag{46}$$

In this case,

$$W = 2\pi R_e^2 E^2 (\frac{R_e^2}{K(\gamma)} - 2\mu), \quad \sigma(R_e) = A - 2\mu E. \tag{47}$$

The release rate of energy

$$G = -\frac{\partial W}{\partial \gamma} = 2\pi R_e^2 E^2 \frac{K'}{K^2} = \pi A^2 \frac{\gamma^2}{l_c} (\frac{1}{L_2} - \frac{1}{L_1}) \tag{48}$$

as the dissipation is

$$D_m = Y_c \int_o^\gamma \frac{r dr}{l_c} = \pi Y_c \frac{\gamma^2}{l_c} \tag{49}$$

We deduce that, during the initiation of the layer A is constant. An indentical answer is obtained when the layer has the maximum thickness l_c. The response of the cylinder with this model of graded damage is exactly the respons of the case obtained by the sharp interface, but the main difference is : the model of sharp interface is not able to described the phase of initiation of damage, unless through complex analysis based on stability and bifurcation [12].

6. CONCLUSIONS

A new approach of damage based on a motion of a thick layer has been proposed, which permits to initiate damage and its evolution with the same constitutive law. The example on a bar shows the influence of the development of the moving layer on the global respons of the system.

The choice of the dissipation process governed by a generalized criterion of Griffith and normality rule provides that the evolution is solution of a variational inequality which allow us to study stability and bifurcation. The generalization of this framework to more complex constitutive equations including plasticitycan be perform in the same form as proposed in [6].

References

[1] G. Allaire, N. Van Goethem, F. Jouve. *A level set method for the numerical simulation of damage evolution*. Rapport 626, Centre de mathématiques appliquées, Ecole polytechnique, 2007.

[2] B. Bourdin, G. Francfort, J.J Marigo. The variational approach to fracture. *J. of Elasticity* 2008; **91**: 5-148.

[3] A. Karma, D. Kessler, H. Levine. Phase-field model of mode III dynamic fracture. *Phys. Rev. Letters* 2001; **8704**(4).

[4] C. Miehe, F. Welschinger, M. Hofacker. Thermodynamically consistent phase field models of fracture: Variational principles and multifield FE implementations. *Int. J. for Numer. Methods in Eng.* 2010; **83**(10): 1273-1311.

[5] N. Moës, C. Stolz, P.E. Bernard, N. Chevaugeon. A level set based model for damage growth : the thick level set approach. *Int. J. for Numer. Methods in Eng.* 2011; **86**(3): 358-380.

[6] R.M. Pradeilles-Duval, C. Stolz. Mechanical transformations and discontinuities along a moving surface. *J. Mech. Phys. Solids* 1995; **43**: 91-121.

[7] R.M. Pradeilles-Duval, C. Stolz. On the evolution of solids in the presence of irreversible phase transformation. *Comptes Rendus Académie des Sciences de Paris, Série II* 1991; **313**: 297-302.

[8] H.D. Bui, K. Dang Van, C. Stolz. Variational-principles applicable to rate boundary-value-problems of elastic-brittle solids with damaged zone. *Comptes Rendus de l'Académie des Sciences, Série II* 1981; **292**: 251-254.

[9] N.Q. Son, R.M. Pradeilles, C. Stolz. On a regularized propagation law in fracture and brittle damage. *Comptes Rendus Académie des Sciences de Paris, Série II* 1989; **309**: 1515-1520.

[10] C. Stolz, N. Moës. A new model of damage: a moving thick layer approach. *Int. J. Fract* 2012; **174**(1): 49-60.

[11] C. Stolz. Thermodynamical description of running discontinuities: Application to friction and wear. *Entropy* 2010; **12**: 1418-1439.

[12] C. Stolz. Bifurcation of equilibrium solutions and defects nucleation. *Int. J. Fract* 2007; **147**: 103-107.